HyperSteiner: Computing Heuristic Hyperbolic Steiner Minimal Trees

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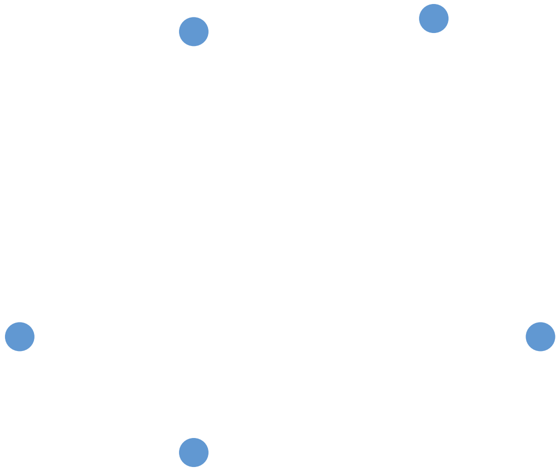
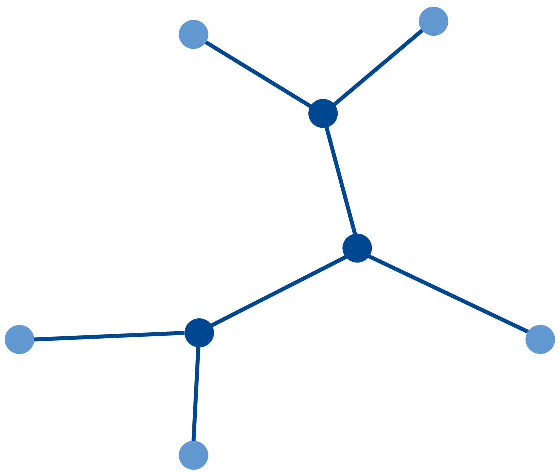
Steiner Minimal Trees

Problem: given a set of n points P in the plane,



Steiner Minimal Trees

Problem: given a set of n points P in the plane, find the set S of points and the tree with vertices P ∪ S minimizing the total edge length.



Applications

SMTs are crucial for (optimal) network design.

Hardness and Heuristics

The problem of obtaining the *exact* SMT is **NP-Hard**

→

Obtain approximate solutions using heuristics

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The **algorithm by Smith et al.** computes *suboptimal* Steiner Minimal Trees by a *divide-and-conquer* approach:

1. Reducing the problem to a local one via the Delaunay triangulation
2. Concatenating local Full Steiner Trees (FSTs)

Non-Euclidean SMTs

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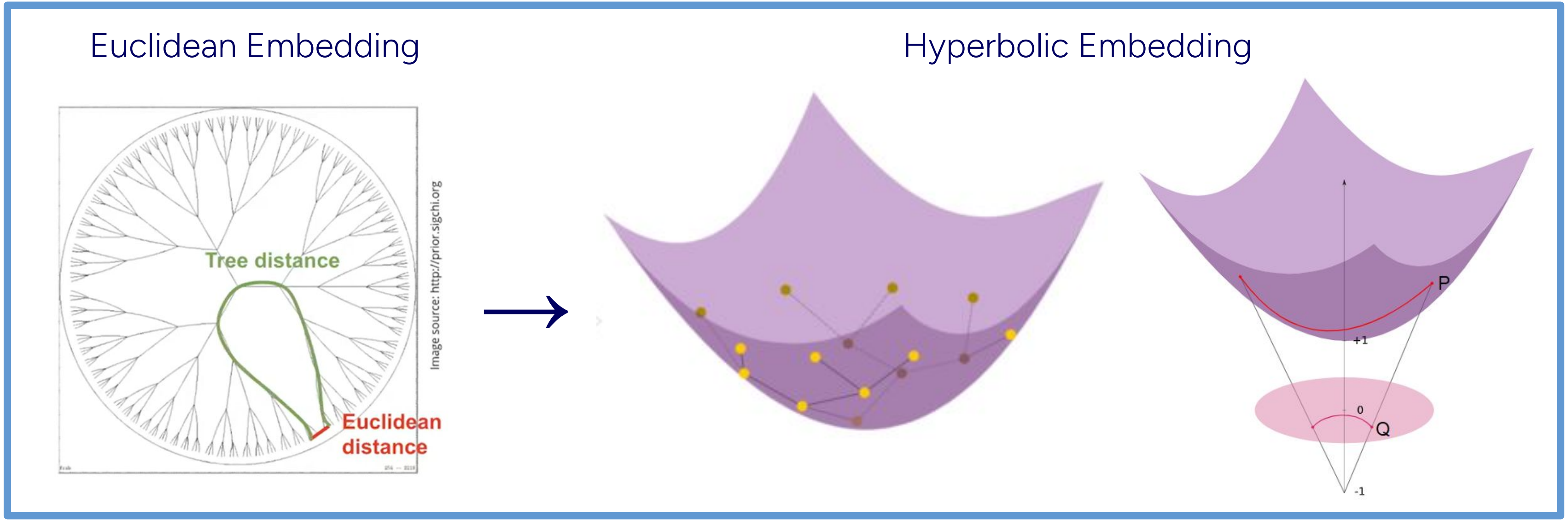
Non-Euclidean SMTs

The SMT problem can be defined on any Riemannian manifold. We focus on the hyperbolic space, since its exponential growth is suitable for representing hierarchical data.



Src: http://summergeometry.org/sgi2021/embedding-hierarchical-data-in-hyperbolic-geometry/

Goals

- Adapt the heuristic method for computing SMTs by Smith et al. to the hyperbolic space
- Apply the method to hierarchy discovery

Smith-Lie-Liebman algorithm

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It has O(n log(n)) computational complexity 🚀

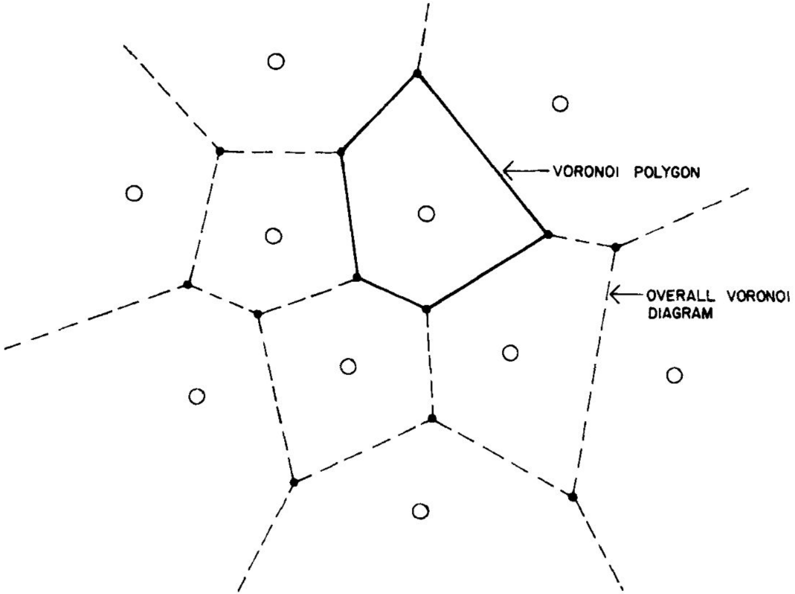
Algorithm 1 SLL Algorithm

Input: Terminals $P \subset \mathbb{R}^2$.

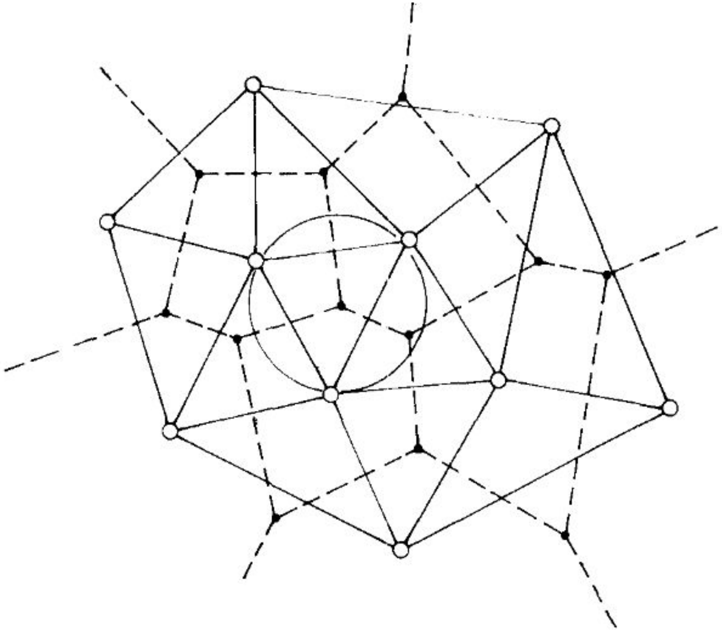
1. Construct the Delaunay triangulation, $\mathrm{DT}(P)$.
2. Construct $\mathrm{MST}(P)$ (Kruskal algorithm) and simultaneously build a priority queue as follows:
   - 2.1. Mark all the triangles $\sigma \in \mathrm{DT}(P)$ containing two edges of $\mathrm{MST}(P)$ and admitting an FST.
   - 2.2. Place the FSTs of marked triangles σ in a queue Q prioritized on $\rho(\sigma)$ (smaller first).
3. Add the FST of the 4-terminal subsets:
   - 3.1. For each marked triangle σ, find its adjacent triangles σ' such that σ and σ' contain three edges of the $\mathrm{MST}(P)$.
   - 3.2. Compute the FST $\sigma \cup \sigma'$ for each of the two possible topologies and add the minimal one to Q.
4. Convert Q to an ordered list and append to it the edges of $\mathrm{MST}(P)$, sorted in non-decreasing order.
5. Let T be an empty tree. An FST in Q is added to T if it does not create a cycle (greedy concatenation).

Output: T – a heuristic SMT of P.

SLL: Delaunay triangulation

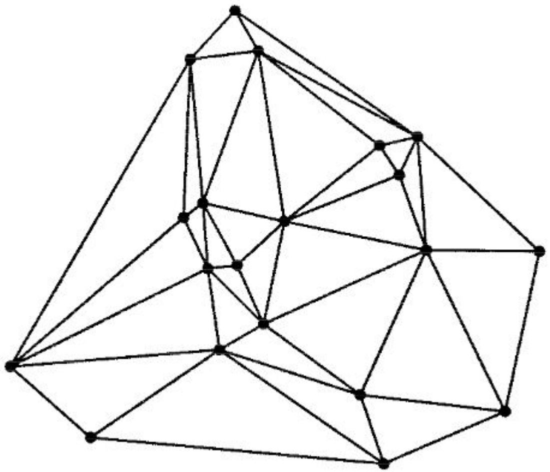


Voronoi diagram



Delaunay triangulation

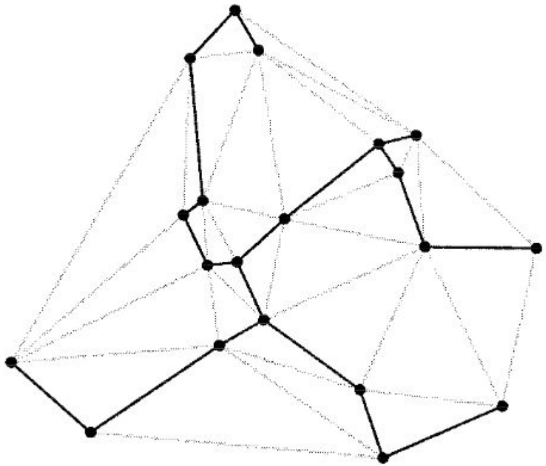
SLL: Minimum Spanning Tree



Delaunay triangulation

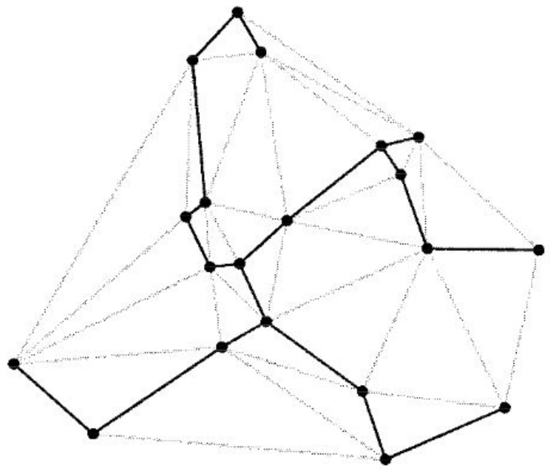
Kruskal

$\Rightarrow$



MST

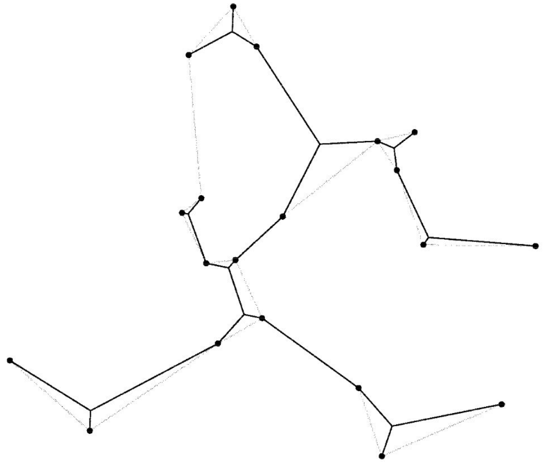
SLL: Heuristic Steiner Minimal Tree



MST

3 and 4 node
Full Steiner Trees

⇒



SMT

Method: Adapting Smith et al.

Goes from local to global solution via Delaunay triangles
⇒ Adaptation from Euclidean to hyperbolic needs:

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Hyperbolic Voronoi Diagram

Hyperbolic Delaunay Triangulation

3 and 4-point Full Steiner Trees in a Hyperbolic model

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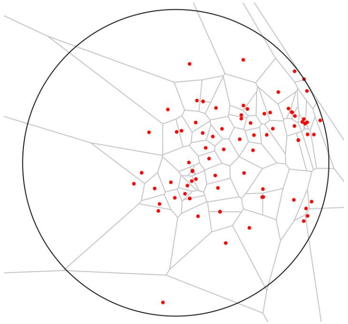

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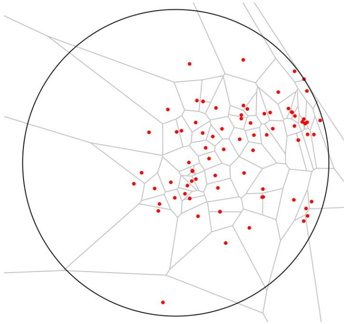

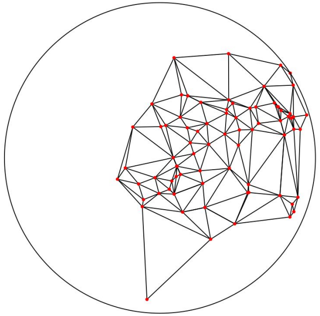

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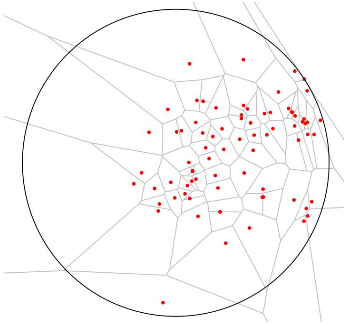
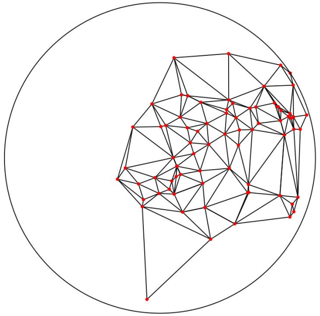
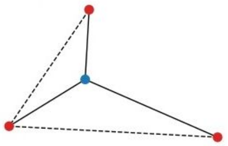
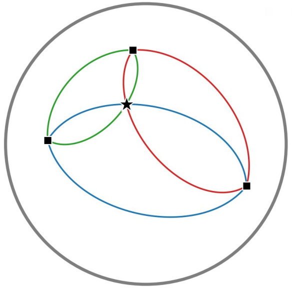




<u>3-points</u>
System of algebraic equations





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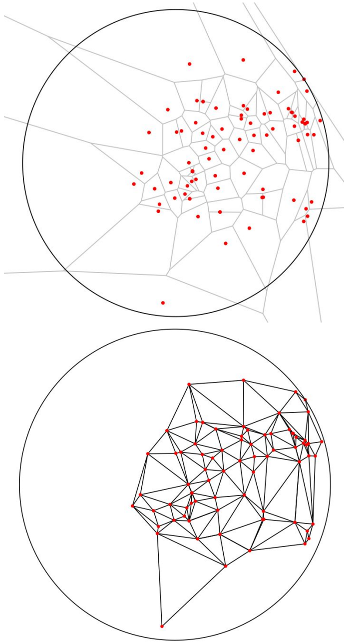

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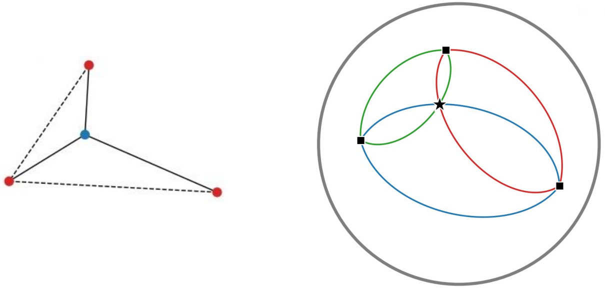
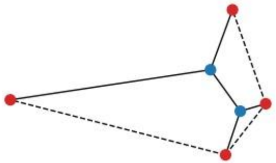


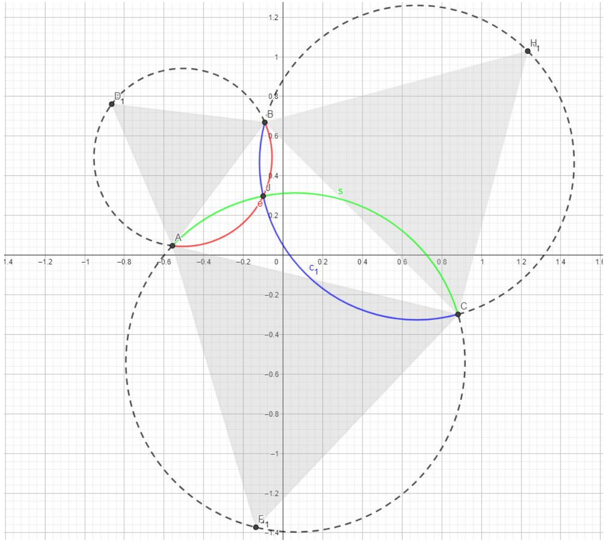
4-points
Approximated recursive algorithm



Method: Solving the 3-point case with isoptic curves

In the 3-point case the Steiner point (if exists) is the locus where all segments of the triangle $\triangle ABC$ are seen under 120° *(i.e., intersection of isoptic curves)*

Euclidean Case (aka Fermat point)



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Hyperbolic Klein model

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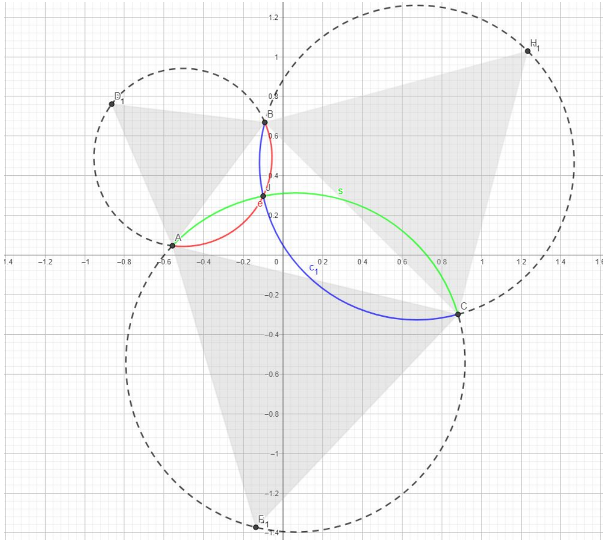

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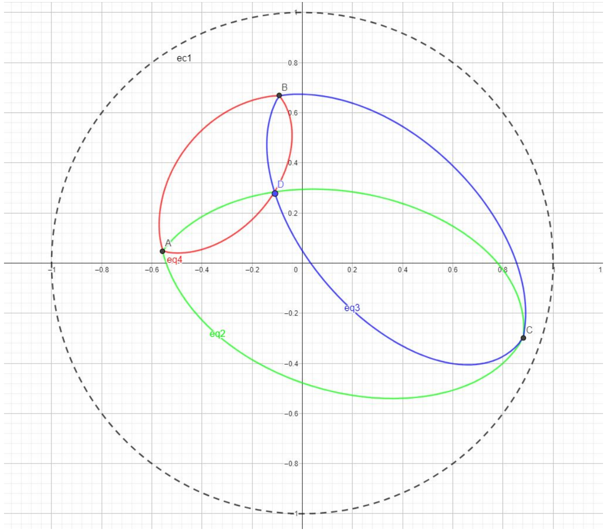

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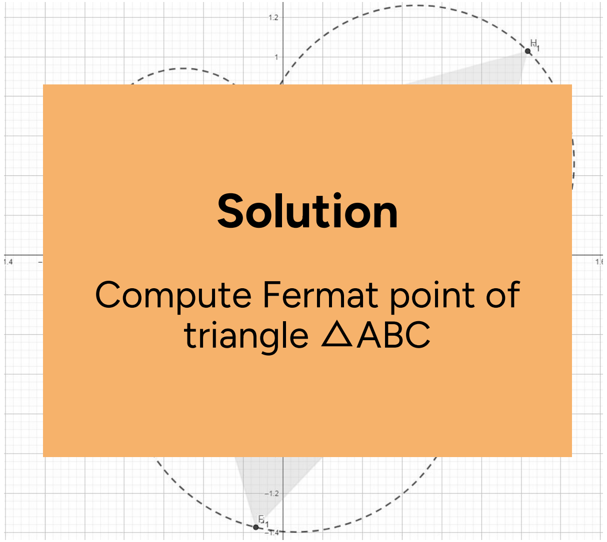
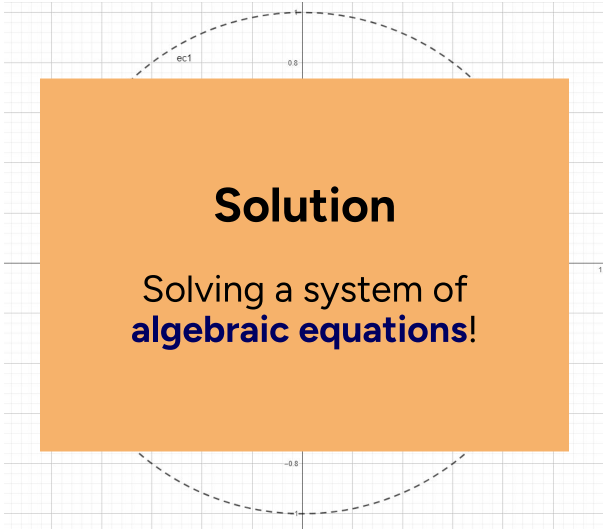


Solution

Compute Fermat point of triangle $\triangle ABC$

→

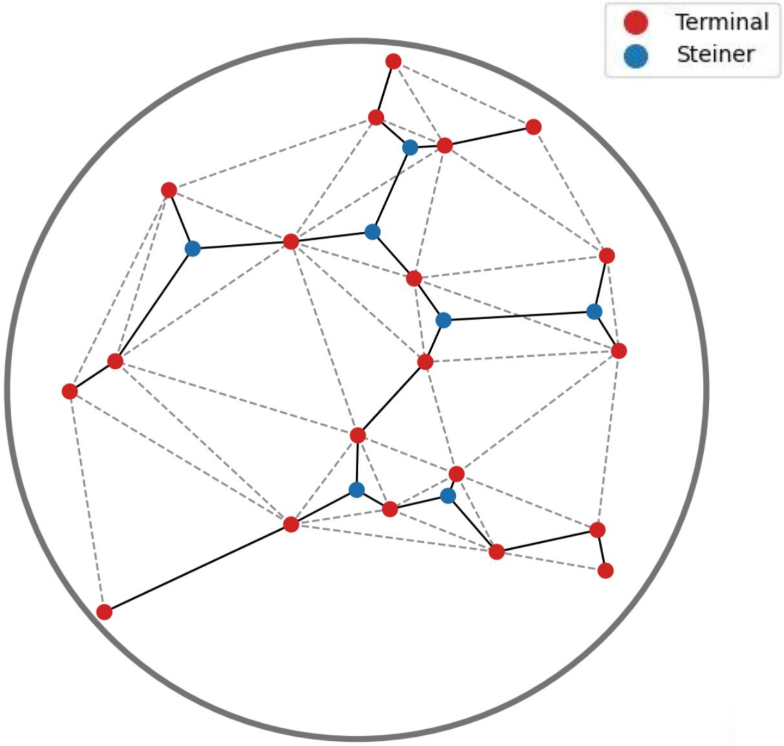
Hyperbolic Klein model



Solution

Solving a system of **algebraic equations**!

Example



Steiner ratio

$$\frac{\text{Steiner tree}}{\text{Minimum Spanning Tree}} = 94\%$$

Experiments

Method validation on synthetic data sampled from different distributions

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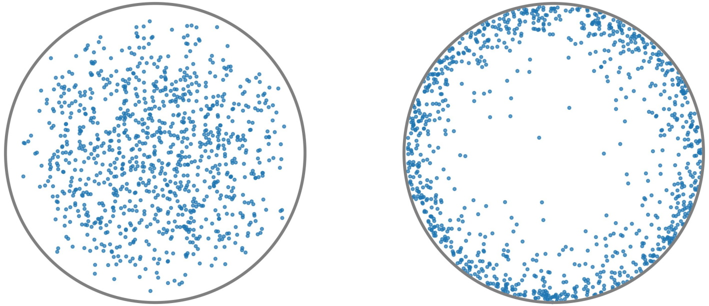
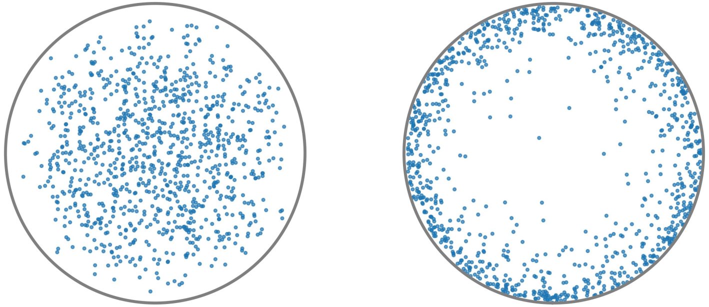


Experiments

Method validation on synthetic data sampled from different distributions

Application to hierarchy discovery on real biological data



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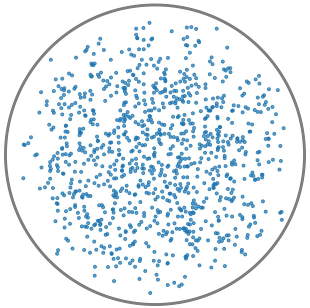

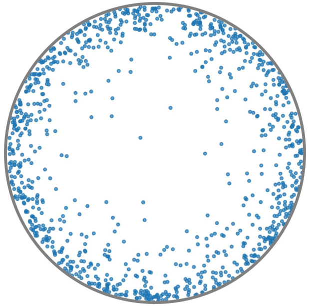

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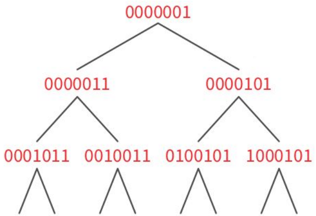
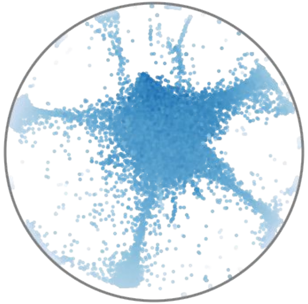
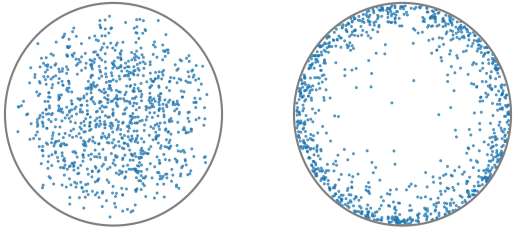




Results for validation on synthetic data



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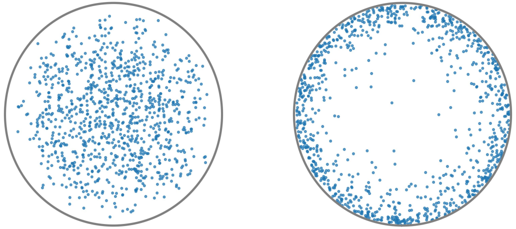

1) HyperSteiner leads to 2-3% improvement over the MST in standard scenarios, similarly to the Euclidean case.

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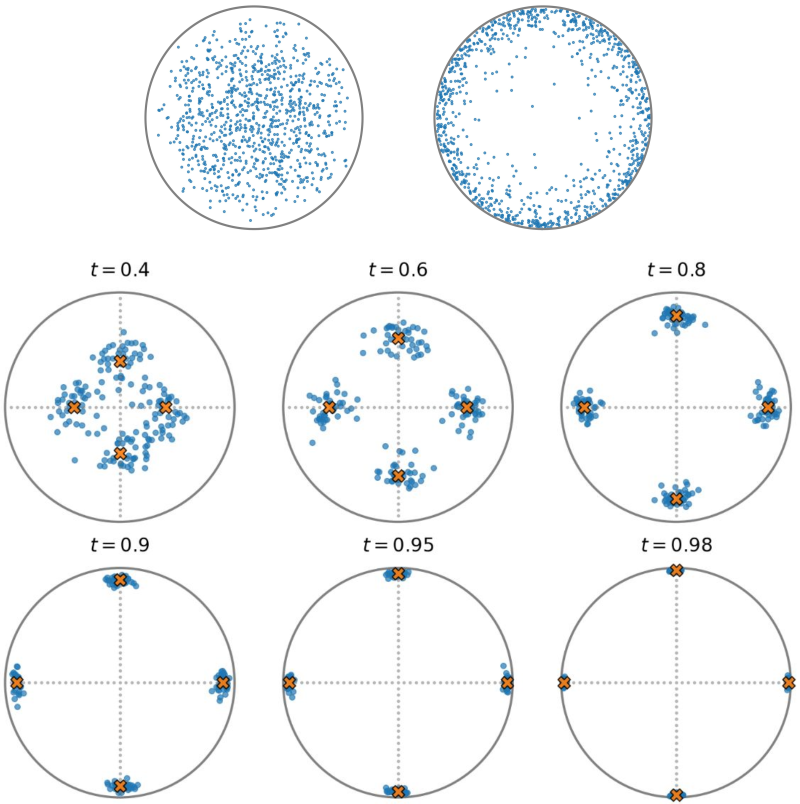

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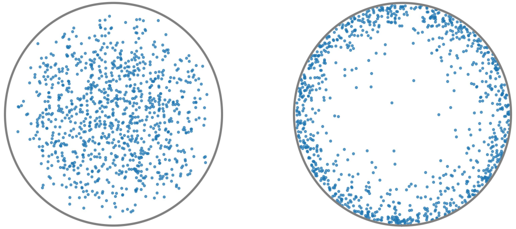

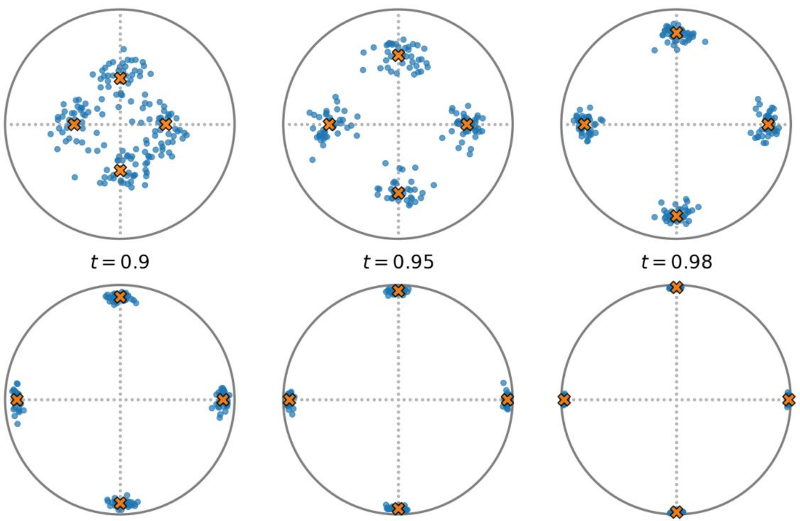

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	$d = 3$		$d = 4$	
t \ $\lvert P\rvert/d$	1	40	1	30
0.40	13.48 ± 0.33	2.52 ± 0.39	9.64 ± 0.27	2.57 ± 0.36
0.60	14.59 ± 0.19	2.33 ± 0.51	11.55 ± 0.17	2.27 ± 0.37
0.80	16.31 ± 0.11	2.44 ± 1.27	14.95 ± 0.11	1.89 ± 0.35
0.90	17.76 ± 0.07	3.16 ± 2.42	18.02 ± 0.08	1.81 ± 0.95
0.95	18.91 ± 0.05	4.47 ± 3.56	20.54 ± 0.06	2.18 ± 1.93
0.98	20.04 ± 0.04	6.63 ± 4.56	23.04 ± 0.05	4.02 ± 4.10

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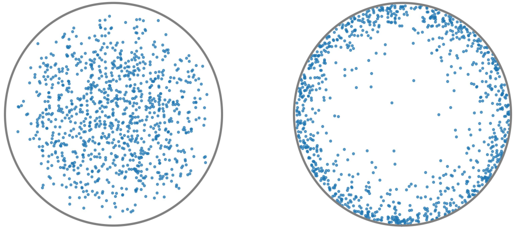

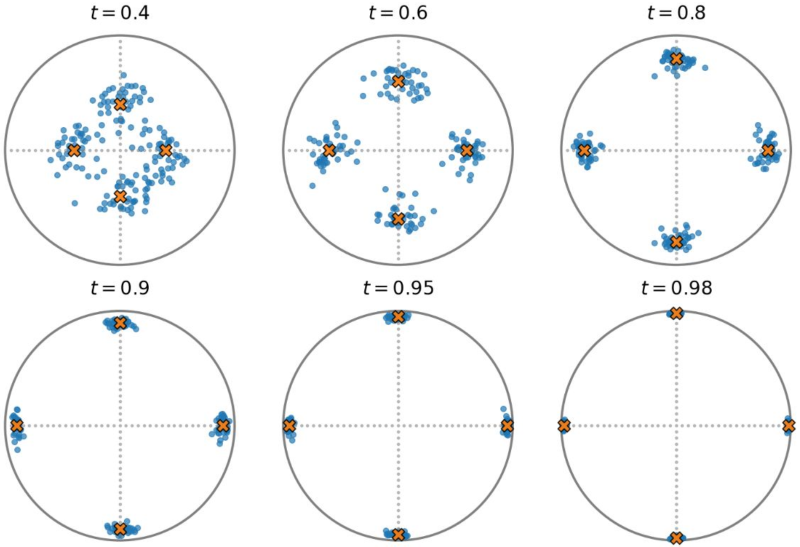

1) HyperSteiner leads to 2-3% improvement over the MST in standard scenarios, similarly to the Euclidean case.

2) HyperSteiner achieves higher improvement over the MST than the maximum theoretical improvement on the Euclidean case (Gilbert-Pollak conjecture ~13.4%).

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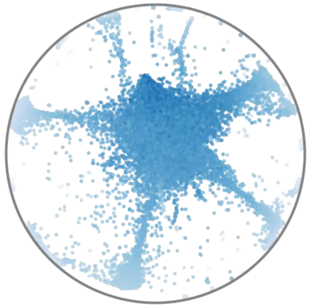
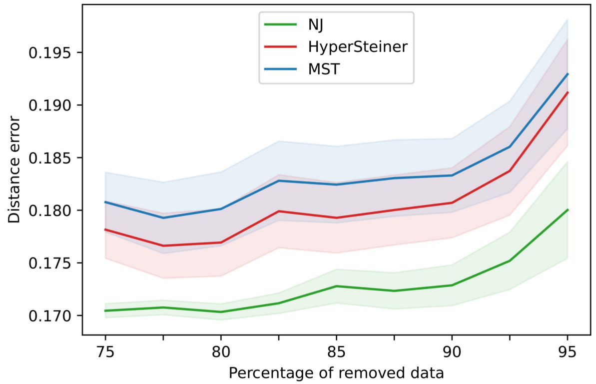
Results for cell age prediction on real data

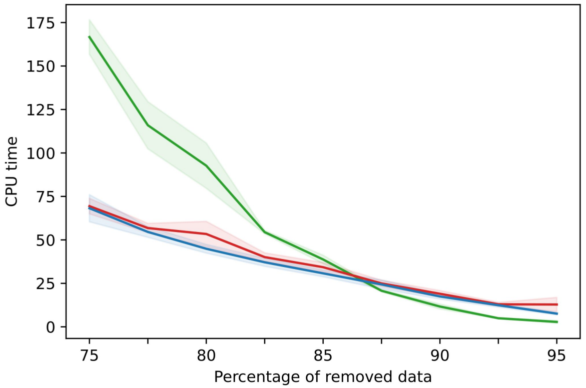
Whole *Planaria* dataset

Method	NJ	HyperSteiner	MST
Distance error	0.17	0.18	0.19
CPU	12604	995	914



Missing data





HyperSteiner:

1) Scales much better than Neighbor Joining to large datasets.

2) Performs better than Minimum Spanning Tree.

Conclusion

We developed a method to compute a heuristic Steiner tree connecting points in a hyperbolic space, with applications to hierarchy discovery in biology.

Questions?